6.4 Recurrence and Transience | 13.4 Le Gall

Return times. Let

$$T_y^0 = 0 \qquad (0^{th} \text{ return time to } y)$$

$$T_y^k = \inf\{r > T_y^{k-1} : X_r = y\}$$

In general $T_y^1 = T_y$ (convention)

$$\rho_{xy} = P_x(T_y < \infty) \quad (\text{interpret?})$$

Theorem: $P_x(T_y^k < \infty) = \rho_{xy}\,\rho_{yy}^{k-1}$

What does this mean?

Pf: Suppose $k \geq 2$. Let

$$Y(\omega) = \begin{cases} 1 & \omega_n = y \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$$

$$=: 1_{\{T_y < \infty\}}(\omega)$$

Let $N = T_y^{k-1}$

Then

$$Y \circ \theta_N = 1_{\{T_y^k < \infty\}}$$

Strong-Markov:

$$E[\, Y \circ \theta_N \mid \mathcal{F}_N] = E_{X_N}[Y] \qquad N < \infty$$

On $N<\infty$, we have $X_N = y$, so

$$E_x[\, Y \circ \theta_N \mid \mathcal{F}_N] = E_y[Y]$$

$$= \mathbb{P}_y(T_y < \infty) = \rho_{yy} \qquad \text{on } N<\infty$$

On $N = \infty$

$$Y \circ \theta_N(\omega) = Y(\Delta) = 0 \qquad (\text{by definition})$$

$$P_x(T_y^k < \infty) = E_x[\, Y \circ \theta_N \, 1_{\{T_y^{k-1} < \infty\}}] \qquad N = T_y^{k-1}$$

Then

$$= E_x[E_x[\, Y \circ \theta_N \mid \mathcal{F}_N] \, 1_{\{T_y^{k-1} < \infty\}}] \quad \text{tower}$$

$$= E_x[\, P_y(T_y < \infty) \, 1_{\{N < \infty\}}]$$

$$= \rho_{yy} \, P_x(T_y^{k-1} < \infty) = \rho_{yy} \, \rho_{xy} \, \rho_{yy}^{k-2}$$

by induction.

$$= \rho_{xy} \, \rho_{yy}^{k-1}$$

Le Gall states :

Let $N_x = \sum_{n=0}^{\infty} 1_{\{X_n = x\}}$ = # of visits.

Prop 13.10

1) If $P_x(T_x < \infty) = 1$ Then $N_x = \infty$ P_x a.s

$\rho_{xx} = 1$

(RECURRENT)

2) If $P_x(T_x < \infty) < 1$ Then $N_x < \infty$ P_x a.s.

(TRANSIENT)

In 2) $P_x(N_x = k) = \rho_{xx}^k = (1-\rho_{xx})\,\rho_{xx}^{k-1}$

(Thus N_x is geometric under k)

$$E_x[N_x] = \frac{1}{P_x(T_x = +\infty)} = \frac{1}{1-\rho_{xx}}$$

Recurrent State: $\rho_{yy} = 1$

Transient State : $\rho_{yy} < 1$

By the theorem $P_y(T_y^k < \infty) = \rho_{yy}^k$

$$\in \begin{cases} 1 & \text{if } y \text{ is recurrent} \\ <1 & \text{if } y \text{ is transient} \end{cases}$$

Note $\{N_y \geq k\} \iff \{T_y^k < \infty\}$

$$E_x N_y = \sum_{k=1}^{\infty} \mathbb{P}_x(N_y \geq k) = \sum_{k=1}^{\infty} \mathbb{P}_x(T_y^k < \infty)$$

$$= \sum_{k=1}^{\infty} \rho_{xy} \rho_{yy}^{k-1}$$

$$= \begin{cases} \infty & \rho_{yy} = 1, \ \rho_{xy} > 0 \\ \dfrac{\rho_{xy}}{1 - \rho_{yy}} & \rho_{yy} < 1 \end{cases}$$

Given some countable state space S, we can break it up into recurrence and transient classes.

$$\boxed{E_x[N_y] = E_x\left[\sum_{k=1}^{\infty} 1_{\{X_k = y\}}\right] = \sum_{k=1}^{\infty} p^k(x,y)}$$

$$= \frac{\rho_{xy}}{1 - \rho_{yy}} \qquad (+\infty \text{ if } \rho_{yy} = 1)$$

Ex: 6.4.3 from Durrett about recurrence of RW.

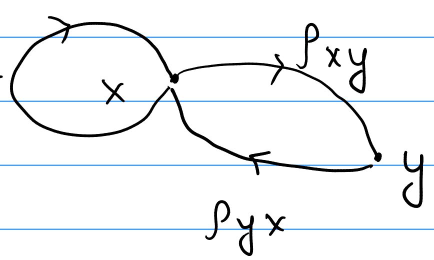
Theorem If x is recurrent and $\rho_{xy} > 0$, then y is recurrent and $\rho_{yx} = 1$.

If there is a chance that you reach y each time you return to x, then you must return from y to x in a finite time. Otherwise x wouldn't be recurrent.

Now, let us start at y. We must go from y to x in finite time. Since x is recurrent, we will keep returning to x; since $\rho_{xy} > 0$, we will definitely return to y after a finite # of recurrences to x.

And the process repeats. So y is recurrent.

recurrence



$\Rightarrow$ In fact if $\rho_{xy} > 0 \Rightarrow \rho_{xy} = 1$ as well by reversing the roles of x and y.

To prove this, Le Gall introduces

$$G(x,y) = E_x[N_y] \quad \text{the Green's function.}$$

The argument is easy. If $\rho_{xy} > 0$ then $\exists\, n_1$ st $p^{n_1}(x,y) > 0$

The difficulty is showing $\rho_{yx} > 0$ to get n_2 st $p^{n_2}(y,x) > 0$

Le Gall repeats an argument similar to mine which is better.

$$0 = P_x(N_x < \infty) \geq P_x\left(T_y < \infty,\ T_x(\theta_{T_y}) = +\infty\right)$$

$$= E_x\left[\mathbb{1}_{\{T_y < \infty\}} \mathbb{1}_{\{T_x = +\infty\}}(\theta_{T_y})\right]$$

$$= E_x\left[E_y\left[\mathbb{1}_{\{T_x = +\infty\}}\right] \mathbb{1}_{\{T_y < \infty\}}\right] \text{(Strong Markov)}$$

$$= P_x(T_y < \infty)\, P_y(T_x = +\infty)$$

$$\Rightarrow P_y(T_x = +\infty) = 0 \quad \Rightarrow \quad \rho_{yx} > 0$$

Thus

$$p^{n_1+n_2+k}(y,y) \geq p^{n_2}(y,x)\, p^k(x,x)\, p^{n_1}(x,y)$$

Sum over k to get

$$\sum p^{n_1+n_2+k}(y,y) \geq p^{n_1}(y,x)\, p^{n_1}(x,y)\, E_x[N_x] = +\infty!$$

Durrett does a bunch of "closed" stuff, I found Le Gall more efficient.

Prop: $G(x,x) = +\infty \iff \rho_{xx} = 1$ (Recurrence)

$$G(x,y) = P_x(T_y < \infty)\, G(y,y) \qquad - \star 1$$

$$= \rho_{xy}\, G(y,y)$$

We proved this more or less. But $\star$ requires SM property:

$$E_x[N_y] = E_x\left[1_{\{T_y<\infty\}}\, N_y \circ \theta_{T_y} \right]$$

This does not count the $t=0$ visit

$$= E_x\left[1_{\{T_y<\infty\}}\, E_y[N_y] \right] = \rho_{xy}\, G(y,y)$$

Types of states

C is closed if $x \in C$ and $\rho_{xy} > 0 \Rightarrow y \in C$

Lemma: If C is closed then $P_x(X_n \in C) = 1 \quad \forall n$

"You start in a closed set, then you never leave it"

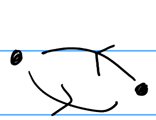
Irreducible: D is irreducible if $x, y \in D \Rightarrow \rho_{xy} > 0$

Lema: Show that if D is irreducible, and $x, y \in D$, then $\exists$ n st $p^n(x,y) > 0$ — all states communicate

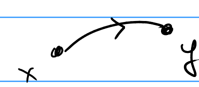
Sometimes also called a "communication class"

A closed state can consist of a union of noncommunicating communication classes.

Ex.



Irreducible



closed

$\{x y\}$ is closed

by $\{x\}$ is not

Theorem: 1) If C is a finite closed set then it has a recurrent state.

2) If it is irreducible, then all states are recurrent

Pf: 2) obviously follows if 1) is true.

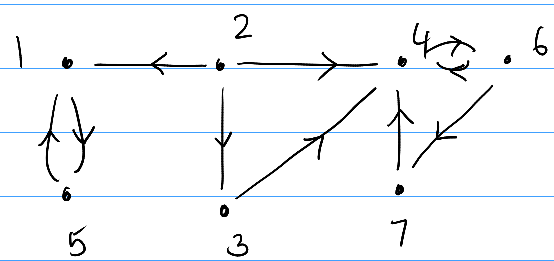
This is because we proved that if $\rho_{xy} > 0$ and x is recurrent Then so is y.

Well if C is finite, then at least one state has to repeat ∞ly often!

$$\sum_{y \in C} N_y = \sum_{y \in C} \sum_{n=1}^{\infty} 1_{\{X_n = y\}} = \sum_{n=1}^{\infty} 1_{\{X_n \in C\}} = +\infty$$

Since C is finite $\Rightarrow \exists\, y$ st $N_y = +\infty$

Ex:



Ex: Is state 2 transient?

$\rho_{12} = 0$ but $\rho_{21} > 0$. If 2 were recurrent, then our previous thm shows, so is 1 and $\rho_{12} > 0$ which is a contradiction.

LESSON: If a class is finite then closed + irreducible
$\Rightarrow$ RECURRENT

Let us completely classify a finite chain:

Take S, it must have at least one
recurrent site x.

Let $C_x = \{y : \rho_{xy} > 0\}$

By the previous 1) $C_x = \{y : \rho_{xy} = 1\}$ and in
fact C_x is recurrent.

$R = \bigcup C_x$ and $S \setminus R$ must be transient

☆ C_x must be irreducible and closed if x is recurrent

LESSON: The recurrent class can be written as
a disjoint union of irreducible and closed
classes

Theorem: (Decomposition) Let $R = \{x : \rho_{xx} = 1\}$ be the set of recurrent sites. Then

$$R = \bigcup_i R_i \quad \text{where each } R_i \text{ is closed}$$

and irreducible.

Pf: $C_x = \{y : \rho_{xy} > 0\}$ (for $x \in R$)

(the set of states that you can reach from x in finite time)

In fact by previous theorem $\rho_{yx} = 1$

and y is also recurrent.

$\Rightarrow \quad C_x \cap C_y = \phi \quad$ OR $\quad C_x = C_y \quad$ (exclusive)

To prove the $C_x \cap C_y = \phi$. Suppose

$\exists$ $x, y\, z$ are states st $z \in C_x \cap C_y$

Given $\rho_{xz} > 0$ and $\rho_{yz} > 0 \Rightarrow \rho_{zy} > 0 = 1$ by our

previous theorem that states if y is recurrent & $\rho_{yz} > 0 \Rightarrow \rho_{zy} = 1$.

So $\exists\, n_1, n_2$ st

$$p_{n_1}(x, z)\, p_{n_2}(z, y) > 0 \quad \Rightarrow \quad p_{n_1+n_2}(x, y) > 0$$

$\Rightarrow$ $\rho_{xy} > 0$. But then this implies $y \in C_x$.

Thus if $\exists\, z' \in C_y \Rightarrow z' \in C_x$ and this shows $C_x = C_y$.

The only other option is C_x and C_y are disjoint.

If S is countably infinite,

closed + irreducible no longer implies recurrence

It would also be transient. Ex: SRW in $d \geq 3$.

Le Gall proves a nicer theorem that not only proves decomposition but also classifies what happens when you start in a transient state.

Thm (13.14) Let S countable.

a) Then $\quad R = \bigsqcup_{i \in I} R_i \quad$ as proved earlier

b) For $i \in I$, and $x \in R_i$, $\quad N_y = +\infty \quad P_x$ a.s. $\quad \forall y \in R_i$

$$N_y = 0 \quad \forall \quad y \in S \setminus R_i$$

Let
$T = \inf\{n : X_n \in R\}$

2 a) $\forall x \in E \setminus R \quad P_x$ a.s $\quad N_y < \infty \quad \forall y \in S \setminus R \quad | \quad T = +\infty$

$\exists$ a random i st $\quad N_y = +\infty \quad \forall y \in R_i \quad | \quad T < \infty$

Pf: Le Gall says $x \sim y$ if $p_{xy} > 0$ is an equivalence relationship on R. (Same argument as above)
R_i corresponds to equivalence classes.

By defn $p_{xy} = 0$ if $y \notin R_i$ and obviously $p_{xy} = 0$ if $y \in S \setminus R$ since if $p_{xy} > 0$ then y is recurrent by previous.

It remains to show $N_y = +\infty \quad \forall y \in R_i \qquad$ We've shown

$$P_y(N_y \geq k+1) = P_y(T_y < \infty) P_y(N_y \geq k) = \rho_{yy} P_y(N_y \geq k)$$

$$= P_y(N_y \geq k) \Rightarrow P_y(N_y \geq k) = 1$$

Same argument applies starting at P_x.

$$P_x(N_y \geq k+1) = P_y(N_y \geq k) = 1$$

$\forall y \in S \setminus R_i$ we've shown $\rho_{xy} = 0 \Rightarrow N_y = 0$ a.s.

2) $x \in S \setminus R$. We've shown $G(x,y) = E_x[N_y] = \dfrac{\rho_{xy}}{1-\rho_{yy}} < \infty$

$\forall y \in S \setminus R. \Rightarrow N_y < \infty$ a.s and one takes a union over y.

On $T < \infty$, what does one do? Let $y \in R_i$

$$P_x(N_y \geq k+1, T < \infty) = P_x(N_y(\theta_T) \geq k+1, T < \infty, X_T \neq y)$$

$$+ P_x(N_y(\theta_T) \geq k, T < \infty, X_T = y)$$

$$= \sum_{z \in R_i, z \neq y} P_z(N_y \geq k+1) P_x(X_T \neq y, T < \infty)$$

$$+ P_y(N_y \geq k) P_x(X_T = y, T < \infty)$$

$$= P_x(X_T \in R_i, T < \infty)$$

$$\Rightarrow P_x(N_y = +\infty, T < \infty) = P_x(X_T \in R_i, T < \infty)$$

$\Rightarrow P_x(\{N_y = +\infty, T < \infty\} \,\Delta\, \{X_T \in R_i, T < \infty\}) = 0 \quad \forall y \in R_i$

and $\Rightarrow$

$$\bigcap_{y \in R_i} \{N_y = +\infty, T < \infty\} = \{X_T \in R_i, T < \infty\}$$

Le Gall does a song and dance about the canonical MC but I ignore it.

<u>Cor</u>: If p is irreducible then every state is recurrent or every state is transient. Also

$$P_x(\cap N_y = \infty) = 1 \quad \text{and} \quad P_x(\cap N_y < \infty) = 1 \qquad \text{resp.}$$

Pf: This follows from $\rho_{xy} > 0 \quad \forall x, y$. The last two follow from previous classification of states thm.

Le Gall then proves the following thm:

Prob 13.17: If X_n is an iid rw with $E|X_1| < \infty$ and $EX_1 = 0$ Then all states are recurrent. $(X_0 = 0)$

Pf: I like this proof because it just uses the machinery we have built.

Assume 0 is transient $U(0,0) < \infty$

$$U(0,x) = \rho_{0x} U(x,x) \Rightarrow U(0,x) \leq U(x,x) = U(0,0)$$

$$\Rightarrow \sum_{|x| \leq n} U(0,x) \leq (2n+1) U(0,0) \leq Cn \quad \text{—} (\star 1)$$

But $\frac{X_n}{n} \to 0$ a.s. and hence $P\left(\frac{|X_n|}{n} \leq \epsilon\right) > \frac{1}{2}$ $\forall n > N$

$$P\left(\frac{|X_n|}{n} \leq \varepsilon\right) = \sum_{|x| \leq \varepsilon n} p_n(0,x) > \frac{1}{2}$$

Now let $m > n$ $\displaystyle\sum_{|x| \leq m\varepsilon} p_n(0,x) \geq \sum_{|x| \leq n\varepsilon} p_n(0,x) > \frac{1}{2}$

since youre summing over more terms.

Then $\displaystyle\sum_{|x| \leq \varepsilon n} U(0,x) = \sum_{m=1}^{n} \sum_{|x| \leq \varepsilon n} p_m(0,x) \geq \sum_{m=N}^{n} \sum_{|x| \leq \varepsilon n} p_m(0,x) \geq \frac{n-N}{2}$

Replace n by εn in $\star 1$. We get $C\varepsilon n \geq \frac{n-N}{2}$, contradiction.

Prop: The chain is irreducible iff $\{x : \mu(x) > 0\}$ generates $\mathbb{Z}$

Note this can be skipped. But note if

$$\mu = \frac{\delta_{-1} + \delta_1}{2}$$ produces 2 noncommunicating recurrence classes.

In previous years I did

1) Branching Process

2) M/G/1 Queue

3) Gambler's ruin, harmonic functions etc.
We have done some of these things using Martingales

4) Recurrence and transience formulas for the birth and death chain

Branching Process

$S = \{0, 1, 2, \ldots\}$

ξ_{ik} = # of children that individual k has in generation n

If $P(\xi_{ik} = 0) = \mu(0) > 0$

X_i = # of individuals in generation i

can be zero.

What is $p(0,k)$?

What is $p(k,0)$? $= \mu(0)^k$

So there are two rels.

$\{0\}$ — Recurrent

$\{1,2,3,\ldots\}$ — ?

☆ Question: Is $\{1,2,3,\ldots\}$ irreducible?
Give me a sufficient condition.

☆ The Branching process has cool properties.

We have shown that
$\{0\}$ is absorbing

You can run off to infinity

1) If $\mu = E[\xi_{ij}] > 1$ (SUPERCRITICAL)

$P(\lim_{k\to\infty} X_k = +\infty) > 0$ (also $P(\lim_{k\to\infty} X_k = 0) > 0$)

2) If $\mu \leq 1$

$P(\lim_{k\to\infty} X_k = 0) = 1$

But in the supercritical case a priori it's unclear if you can persist in some limbo between 0 and $+\infty$.

Any $k \geq 1$ must be transient since if recurrent

$U(k,0) \geq \mu(0)^k > 0 \Rightarrow 0$ is also recurrent and $U(0,k) > 0$
But this is false.

How about the $\mu(0) = 0$ case? Assume $\mu \neq \delta_1$ (BORING)

So $\exists\, k > 1$ st $\mu(k) > 0 \Rightarrow U(k,\ell) > 0$ for $\ell > k$

but $U(\ell,k) = 0 \Rightarrow k$ is transient.

<u>Conclusion</u>: If $\mu(0) > 0$ and if $\mu(0) = 0$ but $\mu \neq \delta_1$ then $\{0\}$ is absorbing and $\{1, 2, \dots\}$ is transient. Our prev. thm then shows $X_n \to 0$ or $X_n \to +\infty$ is a dichotomy.

M/G/1 queue.

A_1 be the # of customers coming in during time interval 1.

$\mu = \sum k a_k = E[A_1]$. Recall $\xi_k = A_k - 1$ serviced customer

$X_n = \{X_{n-1} + \xi_n\}^+$ = # of customers in the queue at time n

So let $S_n = \sum_{i=1}^{n} \xi_i + X_0$

$X_0 + S_n$ and X_n behave the same way until X_n hits 0.

until the stopping time N.

$$N = \{\inf i : X_0 + S_i = 0\}.$$

What's the strategy here?

When $\mu > 1$, you want to be transient, really. So can you show that

$$\mathbb{P}_x (N = +\infty) > 0.$$

where N is the above stopping time. For this, it's enough to show.

$$\lim_{x \to \infty} \mathbb{P}_x (N < \infty) = \mathbb{P} (\inf S_n = -\infty)$$

(The argument is below.)
where $S_n = \sum_{i=1}^{n} \xi_i$

When $\mu > 1$, $\mathbb{E}\xi_m = \mu - 1 > 0$

$\Rightarrow$ $S_n \to \infty$ a.s and

$\inf_n S_n > -\infty$ as

How to show that? Can use Borel-Cantelli.

$\mathbb{P}\left(\left| \frac{S_n}{n} - \mu \right| > t \right) \le e^{-nt}$ Chebyshev.

This is summable. So

$\mathbb{P}(\mu n - t \le S_n \le \mu n + t \ \text{ev}) = 1$

Pick $t < \mu/2$

$\Rightarrow \exists$ a $N(\omega)$ st

$\mu n - nt \le S_n \le \mu n + t \qquad n \ge N(\omega)$

$\Rightarrow \inf_n S_n \ge \inf_{n \le N(\omega)} (S_n, \mu/2 \, N(\omega)) > -\infty$

☆ This is a good exercise,

but maybe it's easiest to just prove?

$$\Rightarrow \quad \mathbb{P}_x\left(\inf_n S_n > -\infty\right) = 1$$

Claim:

$$\lim_{x\to\infty} \mathbb{P}_x(N<\infty) = \mathbb{P}\left(\inf_n S_n = -\infty\right)$$

$$\parallel$$

$$\lim \mathbb{P}\left(S_n + x \le 0 \text{ ev}\right)$$

$$= \lim_{x\to\infty} \mathbb{P}\left(S_n \le -x \text{ ev}\right)$$

$$= \mathbb{P}\left(\inf S_n = -\infty\right) = 0$$

So the chain is transient. since $\mathbb{P}_x(N<\infty)<1$ for large x.

There is a chance that you will never hit 0

For:

$\mu \le 1$. This a lovely stopping time argument

Question: Is X_n a ____ martingale?

Try it

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] \geq \mathbb{E}[X_{n-1} + \xi_n \mid \mathcal{F}_{n-1}]$$

$$= X_n + \mu - 1 \quad \cdots$$

BUT until the stopping time,

$X_{n\wedge N} = X_0 + S_n$ and certainly

$= 0 \qquad n \geq N(\omega)$.

$$\mathbb{E}[X_{n\wedge N} \mid \mathcal{F}_{n-1}] = X_{n-1} \quad \text{on } n<N$$

$$\mathbb{E}[X_N \mid \mathcal{F}_{n-1}] = 0 \leq X_{n-1}$$

of course. Super martingales decrease on average.

Strategy: Use stopping theorem.

<u>Question</u>:

What do we want to do?

we want to control X_n when it gets too large.

$X_{n\wedge N}$ is a non-negative martingale

$\Rightarrow$ STOPPING THM APPLIES.

$\tau =$ 1st time X_n becomes big.

$$T = \inf\{n \geq 1 : X_n \geq M\}$$

$$X_\tau \geq M \text{ on } \{T < N\},$$
$$X_\tau = 0 \text{ on } \{N < T\}$$

$$\Rightarrow \mathbb{E}_x[X_\tau] \leq \mathbb{E}_x[X_0] = x. \quad \left(\begin{array}{c}\text{decrease}\\ \text{on}\\ \text{average}\end{array}\right)$$

$$M\,\mathbb{P}(T<N) + 0.\mathbb{P}(N<T) = x.$$

$$\Rightarrow \mathbb{P}(T<N) = \frac{x}{M} \longrightarrow 0. \quad (\text{as } M \longrightarrow \infty)$$

But clearly $\lim_{M\to\infty} T = +\infty.$ $\Rightarrow \lim_{M\to\infty} P(T<N) = 0$

$$\mathbb{P}(N = +\infty) = 0$$

So the chain is recurrent you keep coming back to 0.

3) Superharmonic / Subharmonic functions

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* Give them branching process as HW. Tell them to Read Williams, chapter 0.

Thm: (6.4.8) Suppose S is countable, reducible with

$$E_x \varphi(X_1) \leq \varphi(x) \text{ for all } x \notin F.$$

and $\varphi(x) = \infty$ as $x \to \infty$.

$\{x : \varphi(x) \leq M\}$ is finite for any $M < \infty$, then the chain is recurrent!

This Martingale technique is very useful.

This submartingale technique used in the M/G/1 queue is quite general.

Thm: S is irreducible and uncountable.

$E_x \varphi(X_1) \leq \varphi(x) \quad \forall \ x \in F$ a given finite set

$\varphi(x) \longrightarrow \infty$ as $x \to \infty$

Then the chain is recurrent.

Pf: let $N = \inf \{ n : X_n \in F \}$

$\varphi(X_{n \wedge N})$ is a submartingale.

$$E[\varphi(X_{n\wedge N}) \mid \mathcal{F}_{n-1}] = E_{X_{n-1}}[\varphi(X_n)] \qquad n \leq \tau$$

$$E[\varphi(X_{n\wedge N}) \mid \mathcal{F}_{n-1}] = \varphi(X_{n-1}) \qquad \tau \leq n-1$$

$E_x[\varphi(X_1)] \leq \varphi(x)$ So this is a supermartingale

let $T_M = \inf \{ n : \varphi(X_n) > M \}$

Since $\{x : \varphi(x) < M\}$ is finite and the chain is irreducible, $T_M < \infty$. So the stopping theorem applies ($\varphi \geq 0$)

$$E_x[\varphi(X_{T_M \wedge N})] \leq E_x[\varphi(X_0)] = \varphi(x) \quad \forall x \notin F$$

As before this is

$$M P(T_M < N) + \sum_{y \in F} \underbrace{\varphi(y) P(X_N = y)}_{+ve} \leq \varphi(x) \quad \forall x \notin F$$

$$= P_x(N = +\infty) = 0$$

$$P_x(N < \infty) = 1 \qquad \forall x \notin F$$

☆ Ex: Prove

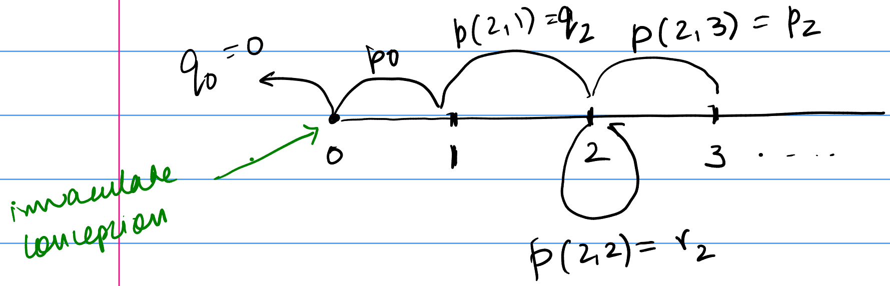
$$P_y(X_n \in F \text{ i.o}) = 1 \qquad \forall y \in S$$

and since F is finite $\exists z \in F$ st

$$P_y(X_n = z \text{ io}) = 1$$

$\Rightarrow$ z is recurrent and since the chain is irreducible, every $y \in S$ is recurrent.

Birth and Death Processes



$S = \{0, 1, 2, \dots\}$ At each time step, a birth or a death happens. This is a generalized 1D RW with a barrier at 0

We want to discover a fn φ st $\varphi(X_n)$ is a Martingale.

$$E[\varphi(X_n) \mid \mathcal{F}_{n-1}] \overset{\text{Markov}}{=} E_{X_{n-1}}[\varphi(X_n)]$$

$$= p_j \varphi(j+1) + q_j \varphi(j-1) + r_j \varphi(j)$$

$$= \varphi(j) \qquad j = 1, 2, \dots$$

— ☆1

And for $j = 0$,

$$E[\varphi(X_n) \mid \mathcal{F}_{n-1}] = p_0 \varphi(1) + r_0 \varphi(0) = \varphi(0)$$

☆2

(☆1) Can be rearranged of course.

$$p_j \varphi(j+1) + q_j \varphi(j-1) = p_j \varphi(j) + q_j \varphi(j)$$

$$\Rightarrow \quad \varphi(j+1) - \varphi(j) = \frac{q_j}{p_j}\left(\varphi(j) - \varphi(j-1)\right) \qquad j = 1, 2, \ldots$$

Unfortunatey ★2 is a bid uselers. We get

$$\varphi(1) = \varphi(0) \quad \Rightarrow \quad \varphi = 0 \quad \forall j$$

so let us drop ★2, and just look at

$$\varphi(X_{n \wedge T_0} \quad , \text{ where } T_0 = \inf\{n \mid X_n = 0\}$$

Then $E[\varphi(X_{n\wedge T_0}) \mid \mathcal{F}_{n-1}] = \varphi(0)$ if $X_{n-1} = 0$

$\Rightarrow n \wedge N < N$

Thus set $\varphi(1) = 1$, $\varphi(0) = 0$ and obtain

$$\varphi(j+1) - \varphi(j) = \prod_{k=1}^{j} \frac{q_k}{p_k}$$

Then $$\varphi(j) = \sum_{m=1}^{j-1} \prod_{k=1}^{m} \frac{q_k}{p_k} + \varphi(1)$$

Does the stopping theorem apply to $\varphi(X_{n\wedge N})$ and the following

let $T_M = \inf\{n \mid X_{n\wedge N} = M\}$ for some $M > 0$?

Yes it does because $\varphi(X_{n\wedge T_0})$ is bounded.

$$\Rightarrow \quad E_x[\varphi(X_{T_0 \wedge T_M})] = \varphi(x)$$

$$\varphi(0) P_x(T_0 < T_M) + \varphi(M) P_x(T_M < T_0) = \varphi(x)$$

Ex: Show that $P_x(T_0 < T_M) + P_x(T_M < T_0) = 1$

$$\Rightarrow \quad P_x(T_0 < T_M) = \frac{\varphi(M) - \varphi(x)}{\varphi(M) - \varphi(0)}$$

Ex: Gamblers ruin. Suppose you play a game where you win a round with prob. p and lose with $q = 1-p$. If you win you gain a dollar and if you lose, you lose a dollar.
The casino has M dollars and you start with x.
Find the probability that you get ruined before the casino gets ruined. (Do both cases $p \neq q$ and $p > q$)

For $M = 10000$, $p = 0.45$, and $p = 0.5$
Plot $P_x(T_0 < T_M)$ as a function of x.

Finally $P_x(T_M < T_0) = \dfrac{\varphi(x) - \cancel{\varphi(0)} \to 0}{\varphi(M) - \cancel{\varphi(0)} \to 0}$

$T_M \geq M - x$ (at least $M-x$ steps)

If $\varphi(M) \longrightarrow \infty$ as $M \to \infty$

$$P_x(T_0 = +\infty) = 0$$

Otherwise

$$\lim_{M\to\infty} P_x(T_0 = +\infty) = \frac{\varphi(x)}{\varphi(\infty)}$$

<u>Theorem</u>: 0 is recurrent iff $\varphi(M) \to \infty$

$$\lim_{M\to\infty} \sum_{m=1}^{\infty} \prod_{j=1}^{m} \frac{q_j}{p_j} = \infty$$

Ex: Figure out $\varphi(M)$ for your Gambler's ruin problem.

Exercises

☆ 5.3.1 (5a) : looks at the positions of X as at the interarrival times. This is a good problem to give when you have done the 2d SRW.

→ Maybe combine with general d SRW.

5.3.2 is already in the text $\rho_{xz} \geq \rho_{xy}\rho_{yz}$

5.3.7 Recurrence iff every superharmonic function is a constant. (Probably uses Thm 5.3.8 which seeks a function φ? Maybe not)

5.3.8 M/M/∞ queue. Good example that uses Theorem 5.3.8 to discover recurrence.